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Some Relationships Between Representation Systems and Physics. II: Non-relativistic Quantum Theory

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Abstract

Using Bunge's procedures, non-relativistic quantum mechanics is discussed. It is shown that if a certain carefully defined idea of determinism is included in the presentation, operators which are not Hermitian need to be used to describe physical processes. An absolute time scale is suggested.

Bunge (1967) has developed an axiomatic procedure which he has used to write down the postulate of special relativistic kinematics. A condensed version of this method is used here to write down a presentation of a large part of non-relativistic quantum theory. The terms FA, SA and PA mean 'formal axiom', 'semantic assumption' and 'physical axiom' respectively. In AX(9), the representation system defined in Part I (Yates, 1968) is used to facilitate the definition of the concept of determinism.

AX(1): Time

(a) T_m is an interval of the real line. FA

(b) T_m is the range of the time function; any $t \in T_m$ represents an instant of time. SA

AX(2): System

(a) P is a non-empty set. FA

(b) Every $P_i \in P$ represents a physical system. SA

AX(3): State of a System

(a) $\{X_i(t) | P_i \in P, t \in T_m\}$ is a set of vectors in a Hilbert space $\boldsymbol{N}$. Each $X_i(t) \in \{X_i(t) | P_i \in P, t \in T_m\}$ is the image of an element of $P \times T_m$, $P \times T_m$ being mapped into $\boldsymbol{N}$ by a mapping f' in such a way that T_m is in one-to-one correspondence with a complete set of basis vectors in $\boldsymbol{N}$. FA

(b) $\{X_t(i) | P_i \in P, t \in T_m\}$ is a set of vectors in a Hilbert space $\boldsymbol{M}$. A mapping g' exists such that each $X_t(i) \in \{X_t(i) | P_i \in P, t \in T_m\}$ is the image of an element of $P \times T_m$. Note that $X_t(i) \neq X_i(t)$.

(c) $\{X(i,t) | P_i \in P, t \in T_m\}$ is a set of vectors in a Hilbert space $\boldsymbol{V}$

such that each $X(i,t) \in \{X(i,t)|P_i \in P, t \in T_m\}$ can be decomposed as follows:

$$X(i,t) = X_t(i) \otimes X_i(t)$$

$$\boldsymbol{V} = \boldsymbol{M} \otimes \boldsymbol{N} \quad \text{FA}$$

(d) Each $X(i,t) \in \{X(i,t)|P_i \in P, t \in T_m\}$ represents the state of a system $P_i \in P$ at time $t \in T_m$. SA

RMK(1): For the present purpose, $\boldsymbol{N}$ will act as an indexing space which would enable one to distinguish say, $X(i,t')$ from $X(i,t'')$ though $X_{t'}(i)$ and $X_{t''}(i)$ may be equal, $t', t'' \in T_m$. A correspondence between each $X_t(i) \in \{X_t(i)|P_i \in P, t \in T_m\}$ and the space co-ordinates of the system $P_i \in P$ at time $t \in T_m$ will be developed in AX(6). $\{X_t(i)|t \in T_m\}$ will need to include at least as much information as the set of Schrodinger descriptions of the state of the system P_i at all times $t \in T_m$. In fact, were it not for AX(9), $\{X_t(i)|t \in T_m\}$ could be identical with this set of descriptions.

As Weyl (1931, p. 49) points out, even in non-relativistic quantum mechanics a system is describable by the use of a co-ordinate system involving time as well as space co-ordinates; if a space-vector description is given in a Hilbert space $\boldsymbol{M}'$ where each physical state corresponds to one ray the space-time description can be given in a tensor product space $\boldsymbol{V}' = \boldsymbol{M}' \otimes \boldsymbol{N}$ without further ado. For instance, the Schrodinger wave-function $\phi(\mathbf{r},t)$ at $t = t'$ could be written in two parts. $\psi_1(\mathbf{r})$ and $\psi_2(t)\delta(t-t')$. $\psi_1(\mathbf{r})$ would be a space-vector description for time t' and $\psi_2(t)\delta(t-t')$ would specify the time. The really important assumption is the idea that one can describe the state of a system by means of a vector in Hilbert space at any time, $t \in T_m$, and as EX(1) will show this assumption is questionable.

It has also been assumed that the ideas 'system' and 'state of a system' are familiar ones. In the context of non-relativistic quantum mechanics, to demand sharper terminology on that matter is to adopt a strongly operationalist point of view, which brings its own drawbacks with it.

AX(4): Observable

(a) Σ is a non-empty set. FA

(b) Corresponding to each $\sigma \in \Sigma$ there exists an $a \in A$. A is a set of linear transformations on $\boldsymbol{M}''$, subspace of $\boldsymbol{M}$. FA

(c) There exists a representation R of $\boldsymbol{M}''$ where the elements of the set

$$\{\bar{X}_t(i)|\bar{X}_t(i) \equiv X_t(i)(\boldsymbol{M}/\boldsymbol{M}''), P_i \in P, t \in T_m\}$$

(which is isomorphic with the set of all vectors in $\boldsymbol{M}''$) are column vectors and the elements of A are square matrices. Each $a \in A$ is Hermitian in R. FA

(d) Each $\sigma \in \Sigma$ is an observable. SA

RMK(2): The concept 'average value of an observable' and similar ideas could now be defined if required.

AX(5): Configuration Space

(a) $\{B_1, \ldots B_n\}$ are non-empty ordered sets of points. FA

(b) $\{B, \ldots B_n\}$ describe a classical n dimensional configuration space. B_1 refers to one dimension, B_2 to another, and so on. SA

RMK(3): Each (n)-tuple in the set $\{\langle b_1 \ldots b_n\rangle | b_1 \in B_1 \ldots b_n \in B_n\}$ describes the co-ordinates of a point in configuration space.

AX(6): Description of a State in Configuration Space

(a) $\{\langle b \ldots b_n\rangle | b_1 \in B_1 \ldots b_n \in B_n\}$ is in one-to-one correspondence with a set of linearly independent basis vectors in $\boldsymbol{M}''$. FA

(b) Every

$$\bar{X}_t(i) \in \{\bar{X}_t(i) | \bar{X}_t(i) \equiv X_t(i)(\boldsymbol{M}/\boldsymbol{M}''), P_i \in P, t \in T_m\}$$

in a function of all the n-tuples

$$\langle b_1 \ldots b_n\rangle \in \{\langle b_1 \ldots b_n\rangle | b_1 \in B_1, \ldots b_n \in B_n\} \quad \text{FA}$$

(c) $X_t(i) \in \{X_t(i) | P_i \in P, t \in T_m\}$ is a description of the disposition of a system $P_i \in P$ in n-dimensional configuration space at time $t \in T_m$. SA

AX(7): Probability

(a) E' is a non-empty set of projections. FA

(b) $e' \in E'$ is a projection on the subspace of $\boldsymbol{V}$ spanned by the basis vectors which (in the case of the configuration space having a Cartesian type co-ordinate system) refer to all points within the n-dimensional parallelotope in n-dimensional configuration space with sides parallel to the axes, and having as co-ordinates for two diagonally opposite points (the diagonal passing through the centre of symmetry), $\langle b_1 \ldots b_n\rangle$ and $\langle b_1' \ldots b_n'\rangle$. Each element of E' corresponds in this way to an element of

$$\{\langle\langle b_1 \ldots b_n\rangle, \langle b_1' \ldots b_n'\rangle\rangle | b_1 \in B_1 \ldots b_n \in B_n, b_1' \in B_1 \ldots b_n' \in B_n\}$$

FA

(c)

$$(e' X(i,t), X(i,t)) \in \{(eX(i,t), X(i,t)) | P_i \in P, t \in T_m, e' \in E'\}$$

is a measure of the probability that the configuration of state $P_i \in P$ at time $t \in T_m$ is describable by some point in the parallelotope indexed as above by points $\langle b_1, \ldots b_n\rangle$ and $\langle b_1', \ldots b_n\rangle$ in configuration space. SA

DF(1): For simplicity deal with only one system P_i from now on. i is retained as an index where necessary or convenient.

AX(8): Physical Process

(a) G' is a non-empty set of transformations on V. FA

(b) $G' = \{G(i,t',t) | (X(i,t') = G(i,t',t).X(i,t);$

$$G(i,t',t'').X(i,t) = 0 \Leftrightarrow t'' \neq t; t,t',t'' \in T_m\} \quad \text{PA}$$

(c) Physical processes are represented by changes where $X(i,t_0)$ is transformed into

$$G(i,t_n,t_{n-1}) \,.\, G(i,t_{n-1},t_{n-2}) \ldots G(i,t_1,t_0) \,.\, X(i,t_0)$$

each $t_0 \ldots t_n \in T_m$, n any integer. SA

AX(9): Determinism

(a) Z_8 is the representation system constructed by Yates (1968) FA

(b) A relationship between Z_8 and V is developed as follows. Statements included in the development of AX(9) are prefixed AX(9).

AX(9)/DF(2.0): T_1 is a denumerable set of times chosen from T_m.

$$T_1^2 = \{\langle t,t'\rangle | t,t' \in T_1\}$$

AX(9)/RMK(4.0): T_1 should include all the values of $t \in T_m$ which are relevant to the physical situation being considered in a particular instance. It is shown in EX(1) that there may be specific restrictions on T_1. The discussion for the case where T_1 is non-denumerable becomes more complicated, and one would probably need to use the methods of continuous model theory (Chang & Keisler, 1966).

*AX(9)/DF(2.1): **Physically relevant*** state vectors are those which correspond to expressions in Z_8, using DF(2.3). ***Real*** state vectors are those $X(i,t)$ defined by AX(3), with $t \in T_1$. *Unreal* state vectors are vectors in V which correspond (by way of DF(2.3)) to expressions in Z_8, but which are not contained in $\{X(i,t) | P_i \in P, t \in T_m\}$.

AX(9)/RMK(4.1): DF(2.3) is constructed in such a way that real state vectors correspond to expressions in T (as defined by AX(3) of Part I, Yates (1968)) whilst unreal state vectors correspond to expressions in R. The only explicit requirement, beyond symmetry and consistency, which Z_8 will impose is the fact that T cannot contain sentences corresponding to processes during which real state vectors are changed by transformations on V to unreal state vectors. Such sentences are either contained in R or do not exist in Z_8. Thus, sentences in T refer to real processes and sentences in R to unreal processes.

AX(9)/DF(2.2): $Q = \{X(i,t) | t \in T_1\}$ is the set of all real state vectors in $\boldsymbol{V}$. $\{X_\varepsilon(i,t) | t \in T_1,\ \varepsilon \in E_0\}$ is the set of all unreal state vectors in $\boldsymbol{V}$.

$$Q \cup \{X_\varepsilon(i,t) | t \in T_1, \varepsilon \in E_0\} = V$$

AX(9)/RMK(4.2): $\{X_\varepsilon(i,t) | t \in T_1, \varepsilon \in E_0\}$ will only be fully defined when the restrictions implied by Z_8 are explicitly imposed, as they are in DF(2.3) and RMK(6).

AX(9)/DF(2.3): There exist sets of operators G, G_A, $G_{A'}$, such that $G_A \cup G_{A'} \cup G = \text{Op}$. E_0 is just the set of indices of the $G_\varepsilon(\cdot,\cdot,\cdot)$ which satisfy the restrictions below,

$$\begin{aligned}
G &= \{G(i,t',t) | G(i,t',t) \,.\, X(i,t) = X(i,t'); \qquad t', t'', t \in T_1; \\
&\qquad G(i,t',t)\, X(i,t'') = 0 \Leftrightarrow t'' \neq t\} \\
G_A &= \{G_\varepsilon(i,t',t) | G_\varepsilon(i,t',t) \,.\, X(i,t) = X_\varepsilon(i,t'); \qquad t'', t', t \in T_1, \varepsilon \in E_0; \\
&\qquad G_\varepsilon(i,t',t) \,.\, X(i,t'') = 0 \Leftrightarrow t'' \neq t\} \\
G_{A'} &= \{G_{\varepsilon_1'}(i,t',t) | G_{\varepsilon_1'}(i,t',t)\, X_\varepsilon(i,t) = X(i,t'); \qquad t'', t', t \in T_1; \\
&\qquad \varepsilon, \varepsilon_1 \in E_0; \qquad G_{\varepsilon_1'}(i,t',t) \,.\, X_\varepsilon(i,t'') = 0 \Leftrightarrow t'' \neq t; \\
&\qquad G_{\varepsilon_1'}(i,t',t) \notin G \cup G_A\}
\end{aligned}$$

Mappings ν, ν_1 from T_1 and $T_1{}^2$ into N are now defined, such that to each $t \in T_1$ corresponds a unique $w \in N$, with the property that $P_A(w) = w$, and to each pair $\langle t,t' \rangle \in T_1{}^2$ corresponds a unique $g(w,w') \in N$, $P_A(w) = w$, $P_A(w') = w'$.

One can also define a one-to-one correspondence μ between G and a subset of E and a many-to-one correspondence μ_1 between $G_A \cup G_A$ and another subset of E. Similarly, one can define many-to-one and one-to-one correspondences μ_1', μ' between subsets of $\boldsymbol{V}$ and E. The range of P_A is also defined. h in the following discussion, is a fixed number in N. $G(i,\cdot,\cdot)$ are symbols for the vector space transformations and $G_1(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are symbols in Z_8.

$$\begin{aligned}
&\nu(t) = w, \qquad P_A(w) = w \\
&\nu(t') = w', \qquad P_A(w') = w' \\
&w' = w \Leftrightarrow t' = t, \qquad t, t' \in T_1 \qquad w, w'' \in N \\
&\mu'(X(i,t)) = G_1(\nu(t), h), \qquad G_1(\nu(t), h) \in T, \qquad t \in T_1 \\
&\mu_1'(X_\varepsilon(i,t)) = G_1(P_\eta(\nu(t)), h) \qquad t \in T_1, \qquad \varepsilon \in E_0 \\
&\nu_1(\langle t', t \rangle) = g(\nu(t'), \nu(t)), \qquad t, t' \in T_1 \\
&\mu(G(i,t',t)) = G_1(\nu(t'), \nu(t)), \qquad t, t' \in T_1 \\
&\mu_1(G_{\varepsilon'}(i,t',t)) = G_1(\nu(t'), P_\eta(\nu(t))), \qquad t, t' \in T_1, \qquad \varepsilon \in E_0 \\
&\mu_1(G_\varepsilon(i,t',t)) = G_1(P_\eta(\nu(t')), \nu(t)), \qquad t, t' \in T_1, \qquad \varepsilon \in E_0 \\
&\mu_1([0]) = \Phi, \qquad P_A(h) = h
\end{aligned}$$

This defines the expressions in Z_8. FA

AX(9)/RMK(4.3): The more general cases where restrictions like $G(i,t',t)X(t'') = 0 \Leftrightarrow t'' \neq t$ are removed from Op involves more complicated mappings but does not appear to change the results obtained.
AX(9)/RMK(4.4): The physical interpretation of $G(i,t,t')$ $(\in G)$ is simply 'when $G(i,t,t')$ operates on a state vector $\mathbf{u}$, a new state vector $X(i,t)$ is produced iff $\mathbf{u} = X(i,t')$. If $\mathbf{u} \neq X(i,t')$ then the resulting expression is not a real state vector, and indeed is not physically relevant'. Now, since $\mu(G(i,t,t')) = G(\nu(t), \nu(t'))$ one can give the same meaning to $G\ (\nu(t), \nu(t'))$, $t, t' \in T_1$, and so on.
AX(9)/RMK(4.41): Fortunately, the explicit binary mappings [ ] of N^2 into N are not required. For our immediate purposes all we need are the truth values. In a longer and possibly more conventional calculus, $\boldsymbol{C}$, one might have written in place of $G(\nu(t), \nu(t'))$ $(= G(w, w')$ say) the expression

$$St(\mathbf{u} \mathbin{\dot\wedge} (p \mathbin{\dot\vee} \neg p)\begin{pmatrix} p \\ p' \end{pmatrix}$$

$$St(a)\begin{pmatrix} b \\ c \end{pmatrix} \qquad \text{means `}c\text{ replaces }b\text{ in expression }a\text{'}$$

$$\begin{array}{lll} \mathbf{u} \mathbin{\dot\wedge} (p \mathbin{\dot\vee} \neg p) = p & \text{iff} & \mathbf{u} = p \\ \mathbf{u} \mathbin{\dot\wedge} (p \mathbin{\dot\vee} \neg p) = \neg p & \text{iff} & \mathbf{u} = \neg p \\ \mathbf{u} \mathbin{\dot\wedge} (p \mathbin{\dot\vee} \neg p) = \phi & \text{iff} & \mathbf{u} \neq p, \mathbf{u} \neq \neg p \end{array}$$

Also, one would require expressions like

- $\exists\ (k)$ Pf (k,p): 'There exists an expression k which proves p.'
- $\exists\ (k')$ Neg $(k', \neg p)$: 'there exists an expression k' which refutes (not p)'
- $\exists\ (k'')$ Pf $(k, \forall\ y \neg$ Pf $(y, \phi))$: 'There exists an expression k'' which proves ϕ can't be proved within $\boldsymbol{C}$.'

In $\boldsymbol{C}$, p would be the symbol referring to $X(i,t)$; $\neg p$ would be the symbol referring to $\{X_\varepsilon(i,t) | \varepsilon \in E_0\}$, and $\mathbf{u}$ would be a variable, for which either a real or unreal state vector would be substituted.

General expressions for the mappings could be developed as in DF(2.2). Similarly, $G([p'',p),[p',p])$ would be replaced by

$$St\left((St(\mathbf{u} \mathbin{\dot\wedge} (p \mathbin{\dot\vee} \neg p))\begin{pmatrix} p \\ p' \end{pmatrix}\right) \mathbin{\dot\wedge} (p' \mathbin{\dot\vee} \neg p'))\begin{pmatrix} p' \\ p'' \end{pmatrix}$$

This form of statement, however, does not immediately indicate any obvious correspondences between $\boldsymbol{C}$ and $\boldsymbol{M}$, nor does it lead to any evident decidability properties, and it needs an unnecessarily complicated axiom system for its statement.

AX(9)/RMK(4.5): Z_8 includes sentences referring to transitions from real to unreal states, but such sentences are contained in R. Note that unless terms like $[t^{(1)}, r^{(1)}]$ $(t^{(1)} \in T_0, r^{(1)} \in R_0)$ are ϕ as required by DF(14) of Part I (Yates, 1968), one could have had sentences contained in T describing transitions from a real state at one time to an unreal state at a later time with a return to a real state at a later time still—a physically unrealistic result. This can be seen by putting $p'' = t^{(1)}$, $p' = r^{(1)}$ and $\mathbf{u} = p = t^{(1)}$ in the last expression of RMK(4.4).

To explain why one uses a product as defined by DF(11) of Part I (Yates, 1968) rather than the product as defined by DF(14), DF(15) of Part I (Yates, 1968) consider the very simple truth value system B containing only two expressions, $g^{-1}(t) \in T$ and $g^{-1}(r) \in R$, but having otherwise the same structure as Z. We determine the binary mapping $[\ ]'$ explicitly, by comparison with the vectors and transformations in $\boldsymbol{V}$. Let the correspondences between vector space descriptions and descriptions in B be indicated by a double arrow ($\Rightarrow$).

The set of vector space descriptions of real states and transitions is called $\check{T}$ and the corresponding set of unreal state and transition descriptions is called $\check{R}$.

For simplicity consider only one time and omit index i. $T_1 = \{t'\}$

$$\{X(t')\} \in \check{T} \Rightarrow G(t, h) \in T$$

$$\{X_\varepsilon(t') | \varepsilon \in E_0\} \in \check{R} \Rightarrow G(r, h) \in R$$

$$\{G_\varepsilon(t', t') | G_\varepsilon(t', t') \,.\, X(t') = X_\varepsilon(t'); G_\varepsilon(t', t')\, X_\varepsilon(t'') = 0 \Leftrightarrow t' \neq t''; \varepsilon \in E_0\} \in \check{R} \Rightarrow G(r, t) \in R$$

$$\{G_{\varepsilon_1'}(t', t') | G_{\varepsilon_1'}(t', t') \,.\, X_\varepsilon(t') = X(t'); G_{\varepsilon_1'}(t', t')\, X_\varepsilon(t'') = 0 \Leftrightarrow t' \neq t''; \varepsilon, \varepsilon_1, \in E_0; G_{\varepsilon_1'}(t', t') \notin \check{T}\} \in \check{R} \Rightarrow G(t, r) \in R$$

Since B contains only two expressions,

$$g(t, t) = t, \qquad g(r, t) = g(t, r) = r \qquad \text{and} \qquad t = h$$

Also

$$\{G(t', t') | G(t', t') \,.\, X(t') = X(t')\} \in \check{T} \Rightarrow G(t, t) \in T$$

Thus

$$\{G(t', t') \,.\, X(t')\} \in \check{T} \Rightarrow G_1([t, t]', [t, h]') \in T$$

So put

$$[t, t]' = t, \qquad [t, h]' = t$$

$$\{G_\varepsilon(t', t') \,.\, X(t')\} \in \check{R} \Rightarrow G_1([r, t]', [t, h]') \in R$$

Therefore

$$[r, t]' = r$$

$$\{G_{\varepsilon'}(t', t') \,.\, X_\varepsilon(t')\} \in \check{R} \Rightarrow G_1([t, r]', [r, t]') = G_1([t, r]', r) \in R \cup \{\Phi\}$$

Thus

$$[t,r]' = t \qquad \text{or} \qquad [t,r]' = \phi$$

Try

$$[t,r]' = t$$

Then

$$\{G(t',t').X_\varepsilon(t')\} \in \check{R} \cup \{\Phi\} \Rightarrow G_1([t,r]',[t,t]') = G_1(t,t) \in T$$

But

$$G_1([t,r]',[t,t]') \in R \cup \{\Phi\}$$

Thus

$$[t,r]' = \phi$$

The result is readily extended to a system such as Z_8 containing many expressions, and shows why $[\cdot, P_A(\cdot)]$ not $[\cdot,\cdot]$ is used for the binary mapping.

AX(9)/RMK(4.5): To illustrate the last remark one can write a truth-value preserving many-to-one correspondence between the set of expressions in Z_8 (though not necessarily Z) and the set of expressions in B, by defining a new function K, which maps

$$\{g(I),\phi | I \subset E, E \subset Z_8\} \qquad \text{onto} \qquad \{t,r,\phi | t,r \in B\}$$

$$K(j) = \phi \Leftrightarrow P_A(j) = \phi, \qquad P_A.P_\eta(j), = \phi, j \in N \cup \{\phi\}$$

$$K(j) = t \Leftrightarrow P_A(j) = j, \qquad j \in N$$

$$K(j) = r \Leftrightarrow P_A.P_\eta(j) = P_\eta(j), \qquad j \in N$$

AX(9)/RMK(4.6): The semantic meaning of what has been said so far in AX(9) can now be written very concisely.

AX(9)(c): The fact that expressions in Z_8 of the type

$$G_1(P_\eta(p'),n) \in R \qquad \text{iff} \qquad G_1(p',n) \in T, \qquad n \in N$$

$$g^{-1}(p), g^{-1}(p') \in P$$

means that the physical process is deterministic. SA

RMK(5): Thus the present calculus, rather than merely affirming that a system changes from the state described by $X(i,t)$ to that described by $X(i,t')$ as one progresses from time t to time t', $(t,t' \in T_1)$ also denies that the system changes to any of the unreal states in the set $\{X_\varepsilon(i,t') | \varepsilon \in E_0\}$.

RMK(6): The product of $\mu_1(G_\varepsilon(i,t,t'))$ and $\mu_1(G_{\varepsilon'}(i,t',t''))$ is $\mu_1([0])$. Thus

$$G_\varepsilon(i,t,t').G_{\varepsilon'}(i,t',t'') = [0], \qquad \varepsilon \in E_0, \qquad t,t',t'' \in T_1$$

Similarly

$$G(i,t,t')\,G_\varepsilon(i,t',t'') = [0],$$

$$G_\varepsilon(i,t,t').G(i,t',t'') = G_{\varepsilon_l}(i,t,t'') \qquad t,t',t'' \in T_1, \qquad \varepsilon,\varepsilon_l \in E_0$$

These equations give the restrictions imposed on G_A, Q and $G_{A'}$ by Z_8. Let

$$H(i,t',t) \,.\, H_{\varepsilon'}(i,t',t) \,.\, H_{\varepsilon}(i,t',t) \qquad t,t' \in T_1, \varepsilon \in E_0$$

be transformations on $\boldsymbol{M}$ and let

$$L(t',t)(t',t \in T_1)$$

be a transformation on $\boldsymbol{N}$, such that

$$\begin{aligned}
&H(i,t',t) \,.\, X_t(i) = X_{t'}(i)\\
&H_{\varepsilon'}(i,t',t) \,.\, X_{t\varepsilon_1}(i) = X_{t'}(i)\\
&H_{\varepsilon}(i,t',t) \,.\, X_t(i) = X_{t'\varepsilon}(i)\\
&X_{t\varepsilon}(i) \otimes X_i(t) = X_{\varepsilon}(i,t)\\
&L(t',t) \,.\, X_i(t) = X_i(t'), \qquad (X_i(t'), X_i(t)) = 0\\
&H(i,t',t) \otimes L(t',t) = G(i,t',t)\\
&H_{\varepsilon'}(i,t',t) \otimes L(t',t) = G_{\varepsilon'}(i,t',t)\\
&H_{\varepsilon}(i,t',t) \otimes L(t',t) = G_{\varepsilon}(i,t',t), \qquad \varepsilon, \varepsilon_1, \in E_0
\end{aligned}$$

As mentioned in RMK(1), $\boldsymbol{N}$ is just being used for indexing purposes. Using the results above

$$\begin{aligned}
&H_{\varepsilon}(i,t,t') \,.\, H_{\varepsilon}(i,t',t'') = [0]\\
&H(i,t,t') \,.\, H_{\varepsilon}(i,t',t'') = [0]\\
&H(i,t,t') \,.\, H(i,t',t'') = H(i,t,t'')\\
&H_{\varepsilon'}(i,t,t') \,.\, H_{\varepsilon'}(i,t',t'') = [0]\\
&H(i,t,t') \,.\, H_{\varepsilon'}(i,t',t'') = [0], \qquad t,t',t'' \in T_1, \qquad \varepsilon \in E_0
\end{aligned}$$

Thus in the matrix representation $H_{\varepsilon}(i,t,t')$ and $H_{\varepsilon'}(i,t,t')$ $(t,t' \in T_1)$ are nilpotent matrices of index two, and $H(i,t,t')$ is of the form

$$\begin{bmatrix} 0 & A(t,t') \\ 0 & B(t,t') \end{bmatrix}$$

where $A(t,t')$ and $B(t,t')$ are yet to be determined.

$$\dim A(t,t') = \dim B(t,t') = \tfrac{1}{2} \dim \boldsymbol{M}$$

RMK(7): A final point, indicating a possibly fruitful line of enquiry, is the physically interesting fact that T^* cannot be represented in any consistent complete extension of Z_8. Thus, there exists no Tarski theory, with an associated consistent complete representation system which is an extension of Z_8 and at the same time contains a predicate capable of representing T^*.

AX(10): Energy

(a) H_i is a transformation on $\boldsymbol{M}$. FA

(b) $\lim\limits_{\delta t \to 0} \{X_{t+\delta t}(i) - X_t(i) + iH_i \,.\, \delta t\, X_t(i)\} = 0, \qquad t \in T_m$ FA

(c) $\mathscr{E}$ is a set of multiples of the identity matrix. FA

(d) If for some $e \in \mathscr{E}$, where $e = EI$, E a scalar,

$$\lim_{\delta t \to 0} \{\overline{X}_{t+\delta t}(i) - \overline{X}_t(i) + ie\,\delta t \overline{X}_t(i)\} = 0$$

then E represents the energy of the system P_i at time $t \in T_m$. SA

EX(1.0): Let system P_i change in state from $X_t(i)$ to $X_{t+\delta t'}(i)$ $(t, t + \delta t \in T_m)$. δt is very small.

$$H(i, t+\delta t, t) \doteqdot H(i, t, t) + \delta t \left(\frac{\partial H(i, t', t)}{\partial t'}\right)_{t'=}$$

$$X_{t+\delta t}(i) = H(i, t+\delta t, t)\, X_t(i)$$

$$\doteqdot H(i, t, t)\, X_t(i) + \delta t \left(\frac{\partial H(i, t', t)}{\partial t'}\right)_{t=t'} X_t(i)$$

Let $\delta t \to 0$. Then

$$-iH_i = \left(\frac{\partial H(i, t', t)}{\partial t'}\right)_{t'=t}$$

$$= \begin{bmatrix} 0 & (\partial A(i, t', t)/\partial t)(t = t') \\ 0 & (\partial B(i, t', t)/\partial t')(t = t') \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -ia \\ 0 & -i\hat{E} \end{bmatrix}$$

Assume H_i is linear and independent of time. a and $\hat{E}$ are now time-independent matrices. If one had wished to make H_i time-dependent, or indeed to split H_i into parts and use a perturbation technique (Weyl, 1931, p. 30), the development is more complicated but a similar kind of process can be carried out.

Solving the equation in AX(10)(b) gives

$$X_t(i) = \exp[-iH_i(t - t')]\, X_{t'}(i)$$

$$= \begin{bmatrix} 1 & a\{\exp[-i\hat{E}(t - t')] - 1\}/\hat{E} \\ 0 & \exp[-i\hat{E}(t - t')] \end{bmatrix} X_{t'}(i)$$

for $\exp[-iH_i(t - t')]$ to satisfy the requirements imposed by Z_8, if $t, t' \in T_1$. $T_1 \subset T_m$ is the denumerable set of times which are relevant to the physical problem in hand.

$$\exp[-iH_i(t - t')]\, X_{t'}(i) = \begin{bmatrix} 0 & A(i, t, t') \\ 0 & B(i, t, t') \end{bmatrix} X_{t'}(i)$$

This is true when, for some f

$$\begin{bmatrix} 1 & a\{\exp[-i\hat{E}(t - t')] - 1\}/\hat{E} \\ 0 & \exp[-i\hat{E}(t - t')] \end{bmatrix} X_{t'}(i) = \begin{bmatrix} 0 & f(t, t') \\ 0 & \exp[-i\hat{E}(t - t')] \end{bmatrix} X_{t'}(i)$$

Let $X_{t'}(i)$ be written as

$$\begin{bmatrix} X_{t'1}(i) \\ X_{t'2}(i) \end{bmatrix}$$

where $X_{t'1}(i)$ and $X_{t'2}(i)$ are vectors in spaces P_1, P_2. P_1 and P_2 are each of dimension $\frac{1}{2}\dim\boldsymbol{M}$, and $\boldsymbol{M} = P_1 \oplus P_2$.

EX(1.1): $f(t,t')$ can be evaluated as follows

$$f(t,t')\,X_{t'2}(i) = X_{t'1}(i) + a\{\exp[-i\hat{E}(t-t')] - 1\}\,X_{t'2}(i)/\hat{E}$$

Using Z_8, matrices $H_\varepsilon(i,t,t')$ and $H_{\varepsilon'}(i,t,t')$ are applied to a vector $X_{t'\varepsilon}(i)$ with $\varepsilon \in E_0$ and $t,\, t' \in T_1$.

$$H_\varepsilon(i,t,t')\,X_{t'\varepsilon}(i) = \begin{bmatrix} 0 \end{bmatrix}$$

$$H_{\varepsilon'}(i,t,t')\,X_{t'\varepsilon'} = \begin{bmatrix} 0 \end{bmatrix}$$

Thus

$$X_{t'\varepsilon}(i) = \begin{bmatrix} X_{t'\varepsilon 1}(i) \\ 0 \end{bmatrix}$$

$X_{t'\varepsilon 1}(i)$ is a vector in P_1. Thus since the choice of $H_{\varepsilon'}(i,t,t')$ ($\varepsilon \in E_0$ is restricted only by the requirements of RMK(6), vectors in

$$\{X_{t\varepsilon}(i) | t \in T_1\}$$

have no components in P_2.

Further, $\{X_{t\varepsilon}(i) | t \in T_1, \varepsilon \in E_0\}$ contains all the vectors in P_1. Thus, since $\{X_{t\varepsilon}(i) | t \in T_1, \varepsilon \in E_0\}$ and Q must be disjoint because of the consistency of Z_8, $X_t(i)$ can be written, for each $t \in T_1$, as

$$\begin{bmatrix} 0 \\ X_{t2}(i) \end{bmatrix}$$

where $X_{t2}(i)$ is a vector in P_2.

Thus $f(t,t') = a\{\exp[-i\hat{E}(t-t')] - 1\}/\hat{E} = 0$.

Thus $\exp[-i\hat{E}(t-t')] = 1$, for all $t,\, t' \in T_1$.

EX(1.2): AX(4), AX(5) and AX(6) make it natural to suppose that $X_{t2}(i)$ is a vector in $\boldsymbol{M}''$, and that $\boldsymbol{M}''$ and P_2 are at least not disjoint. If the restriction (DF(1)) that only a single system P_i is dealt with had not been made, then a more rigorous relationship between $\boldsymbol{M}''$ and P_2 could be developed in a straightforward but lengthy manner, though difficulties occur in any situation where the conventional superselection rules arise (Schweber, 1961).

EX(1.3): The argument of EX(1.1) shows that the allowed values of $t \in T_1$ are integral multiples of $2\pi/\hat{E}$. This simply means that it is only at such times both that the conditions imposed by Z_8 are fulfilled, and that suitable transformations exist in Hilbert space. At times $t = 2n\pi/\hat{E}$, $t \in T_m$, one obtains real 'expectation values' for H_i and H satisfies the requirement for membership of A given by AX(4).

Also, $(X_t(i), X_t(i))$ oscillates as t varies. On the basis of the present nonrelativistic model such oscillations are not observable. It may be possible to obtain analogous results for a quantum field model by use of axiomatic formalism (Haag & Kastler, 1964) and algebraic logic (Halmos, 1962); such a model would seem more likely to lead to experimental comparisons.

A number of arguments have already appeared in the literature supporting the use of a timescale where the time parameter is restricted to a denumerable set of equally spaced values. The 'absolute time' is usually related probabilistically to a 'physical time' (Rankin, 1965).

An alternative would be to postulate a 'cosmic time' with time $T = 0$ set at some definite instant in the past. Then the restriction would be on $\hat{E}$, so that $\hat{E} = 2n\pi/T$. n would be such a large integer that this energy quantization could be very difficult to observe.

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References

Bunge, M. (1967). *Review of Modern Physics*, **39**, 463.

Chang, G. C. and Keisler, H. J. (1966). *Continuous Model Theory*. Princeton University Press.

Haag, R. and Kastler, D. (1964). *Journal of Mathematical Physics*, **5**, (7), 848–861.

Halmos, P. R. (1962). *Algebraic Logic*. Chelsea.

Rankin, B. (1965). *Journal of Mathematical Physics*, **6**, (7), 1057–61.

Schweber, S. S. (1961). *Relativistic Quantum Field Theory*, p. 5. Harper.

Thouless, D. J. (1961). *The Quantum Mechanics of Many-Body Systems*, p. 36. Academic Press.

Weyl, H. (1931). *The Theory of Groups and Quantum Mechanics*. Dover reprint.

Yates, J. (1968). *International Journal of Theoretical Physics*, Vol. 1, No. 2.